

CONVERGENCE OF KÄHLER-RICCI FLOW WITH INTEGRAL CURVATURE BOUND

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ABSTRACT. Let $g(t)$, $t \in [0, +\infty)$, be a solution of the normalized Kähler-Ricci flow on a compact Kähler n -manifold M with $c_1(M) > 0$ and initial metric $g(0) \in 2\pi c_1(M)$. If there is a constant C independent of t such that

$$\int_M |Rm(g(t))|^n dv_t \leq C,$$

then, for any $t_k \rightarrow \infty$, a subsequence of $(M, g(t_k))$ converges to a compact orbifold (X, h) with only finite many singular points $\{q_j\}$ in the Gromov-Hausdorff sense, where h is a Kähler metric on $X \setminus \{q_j\}$ satisfying the Kähler-Ricci soliton equation, i.e. there is a smooth function f such that

$$Ric(h) - h = \nabla \bar{\nabla} f, \quad \text{and} \quad \nabla \nabla f = \bar{\nabla} \bar{\nabla} f = 0.$$

1. INTRODUCTION

On a compact Kähler n -manifold M with $c_1(M) > 0$, the normalized Kähler-Ricci flow equation is

$$(1.1) \quad \partial_t g(t) = -Ric(g(t)) + g(t) = \sqrt{-1} \partial \bar{\partial} u_t,$$

for a family of Kähler metrics $g(t) \in 2\pi c_1(M)$, where we identify Kähler metrics with the Kähler forms. In [6], it is proved that a solution $g(t)$ of (1.1) exists for all times $t \in [0, \infty)$. Perelman (cf. [19]) has proved some important properties for the solution $g(t)$, $t \in [0, \infty)$, of (1.1): there exist constants $C > 0$ $\kappa > 0$ independent of t such that

- (1) $|R(g(t))| < C$, and $\text{diam}_{g(t)}(M) < C$,
- (2) $|u_t|_{C^1(g(t))} < C$,
- (3) $(M, g(t))$ is κ -noncollapsed, i.e. for any $r < 1$, if $|R(g(t))| \leq r^{-2}$ on a metric ball $B_{g(t)}(x, r)$, then

$$(1.2) \quad \text{Vol}_{g(t)}(B_{g(t)}(x, r)) \geq \kappa r^{2n}.$$

By assuming that the Ricci curvature is uniformly bounded along the flow, Sesum and Tian have proved that, for any sequence of times $t_k \rightarrow \infty$, a subsequence of $(M, g(t_k + t))$ converges to $(X, g_\infty(t))$, where X is smooth outside a singular set, and $g_\infty(t)$ satisfies the Kähler-Ricci soliton equation (cf. [19] and [17]). In a recent preprint [18], Sesum has proved that X is actually a Kähler manifold if $n \geq 3$, and $g(t)$ satisfies an additional

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integral bound of curvature operators. The purpose of this note is to study the convergence of $(M, g(t_k))$ by assuming an integral bound of curvature operators instead of the uniform bound for Ricci curvatures.

Theorem 1.1. *Let $g(t)$, $t \in [0, +\infty)$, be a solution of the normalized Kähler-Ricci flow (1.1) on a compact Kähler n -manifold M with $c_1(M) > 0$ and initial metric $g(0) \in 2\pi c_1(M)$. If there is a constant C independent of t such that*

$$\int_M |Rm(g(t))|^n dv_t \leq C,$$

then, for any $t_k \rightarrow \infty$, a subsequence of $(M, g(t_k))$ converges to a compact orbifold (X, h) with only finite many singular points $\{q_j\}$ in the Gromov-Hausdorff sense. Furthermore, on $X \setminus \{q_j\}$, h is a Kähler metric satisfying the Kähler-Ricci soliton equation, i.e. there is a smooth function f such that

$$Ric(h) - h = \nabla \bar{\nabla} f, \quad \text{and} \quad \nabla \nabla f = \bar{\nabla} \bar{\nabla} f = 0.$$

Here, we call a topological space X an orbifold if X is a smooth manifold outside a finite set of singular points $\{q_j\}$, and there is a neighborhood around every singular point homeomorphic to a cone on a spherical space form $C(S^{2n-1}/\Gamma)$, $\Gamma \subset SO(2n)$. A metric h on X is a Riemannian metric on $X \setminus \{q_j\}$, and, in a local uniformization $B^{2n} \setminus \{0\}$, h extends to a C^0 -metric on the ball B^{2n} . Note that this definition is different from the one in [3], which allows several spherical cones joint at a single vertex.

In [14], Perelman claimed that, if M admits a Kähler-Einstein metric with positive scalar curvature, then a solution of (1.1) converges to the Kähler-Einstein metric. In the case of M admitting a shrinking Kähler-Ricci soliton, Tian and Zhu obtained the same result in [22]. However, there are Kähler manifolds with $c_1 > 0$ admitting no Kähler-Einstein metrics and no holomorphic vector field, thus no Kähler-Ricci soliton ([20]). In this situation, solutions of (1.1) will develop singularities when times tend to infinity by Theorem 1.1 and [17].

If $n = 2$, the L^2 -norm of the curvature operator of a Kähler metric is uniformly bounded by terms of the first and the second Chern class and its Kähler class (c.f. [16] and [7]). In this case, Sesum has claimed a strong version of Theorem 1.1 basing on an unpublished work on the Kähler-Ricci flow due to Tian (c.f. [16]).

Theorem 1.2 (Sesum, Tian). *Let $g(t)$, $t \in [0, +\infty)$, be a solution of the normalized Kähler-Ricci flow (1.1) on a compact Kähler surface M with $c_1(M) > 0$ and initial metric $g(0) \in 2\pi c_1(M)$. Then, for any $t_k \rightarrow \infty$, a subsequence of $(M, g(t_k))$ converges to an orbifold (X, h) with only finite many singular points $\{q_j\}$ in the Gromov-Hausdorff sense. Furthermore, on $X \setminus \{q_j\}$, h is a Kähler metric satisfying the Kähler-Ricci soliton equation.*

The organization of the paper is as follows: In §2, we give an estimate for the harmonic radius of a solution $g(t)$ of the normalized Kähler-Ricci flow (1.1), which plays a central role in the proof of Theorem 1.1. Then we prove Theorem 1.1 in §3.

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2. MAIN ESTIMATES

If (M, g) is a complete Riemannian manifold, for a fixed constant $\Lambda > 1$, the *harmonic radius* $r_h(g)(x)$ at $x \in M$ is the largest radius of a geodesic ball centered at x on which there are harmonic coordinates $\{h_i\}$ such that, if $g_{ij} = g(\nabla h_i, \nabla h_j)$, then $g_{ij} = \delta_{ij}$ and

$$\Lambda^{-1} \cdot I \leq (g_{ij}) \leq \Lambda \cdot I,$$

$$r_h(g)^{1+\alpha} \|g_{ij}\|_{C^{1,\alpha}} \leq \Lambda,$$

on $B_g(x, r_h(g)(x))$ (cf. [2]). In [2], a lower bound of $r_h(g)$ is obtained by assuming a uniform bound for Ricci curvature, and a small L^n -norm for curvature operator. The goal of this section is to generalize this result to the solution of the normalized Kähler-Ricci flow (1.1).

Let $g(t)$, $t \in [0, +\infty)$, be a solution of the normalized Kähler-Ricci flow (1.1) on a compact Kähler n -manifold M with $c_1(M) > 0$ such that $g(0) \in 2\pi c_1(M)$. Recall that Perelman has shown that there are constants C and κ independent of t such that

$$(2.1) \quad |R(g(t))| \leq C,$$

and $g(t)$ is κ -noncollapsed, i.e. for any $r < 1$, if $|R(g(t))| \leq r^{-2}$ on a metric ball $B_{g(t)}(x, r)$, then

$$(2.2) \quad \text{Vol}_{g(t)}(B_{g(t)}(x, r)) \geq \kappa r^{2n},$$

(cf. [19]).

Proposition 2.1. *There are constants $\varepsilon > 0$ and $\overline{C} > 0$ independent of t such that, for any $t \in [1, +\infty)$, if, on a metric ball $B_{g(t)}(x, 2r)$,*

$$\int_{B_{g(t)}(x, 2r)} |Rm(g(t))|^n dv_t \leq \varepsilon,$$

then the harmonic radius $r_h(g(t))$ satisfies

$$\inf_{B_{g(t)}(x, \frac{r}{2})} r_h(g(t)) \geq \overline{C}r.$$

Proof. Actually, we should prove that there are constants $\varepsilon > 0$ and $\overline{C} > 0$ independent of t such that, for any $t \in [1, +\infty)$, if, on a metric ball $B_{g(t)}(x, 2r)$,

$$(2.3) \quad \int_{B_{g(t)}(x, 2r)} |Rm(g(t))|^n dv_t \leq \varepsilon,$$

then

$$(2.4) \quad \frac{r_h(g(t))(y)}{d_{g(t)}(y, \partial B_{g(t)}(x, r))} \geq \overline{C},$$

where $d_{g(t)}(y, \partial B_{g(t)}(x, r)) = \text{dist}_{g(t)}(y, \partial B_{g(t)}(x, r))$. If it is not true, there is a sequence of times $\{t_k\}$, and two sequences of points $\{x_k\}$, $\{y_k\}$ such that

$$\frac{r_h(g_k)(y)}{d_{g_k}(y, \partial B_k)} \geq \frac{r_h(g_k)(y_k)}{d_{g_k}(y_k, \partial B_k)} \longrightarrow 0,$$

for all $y \in B_k$, when $k \longrightarrow \infty$, where $g_k = g(t_k)$ and $B_k = B_{g_k}(x_k, r)$, but

$$\int_{B_{g_k}(x_k, 2r)} |Rm(g_k)|^n dv_k \leq \varepsilon.$$

If $\mu_k = r_h^{-2}(g_k)(y_k)$ and $\tilde{g}_k = \mu_k g_k$, then $r_h(\tilde{g}_k)(y_k) = 1$,

$$|R(\tilde{g}_k)| = \mu_k^{-1} |R(g_k)| \leq \frac{C}{\mu_k} \longrightarrow 0, \quad \text{by (2.1), and}$$

$$d_{\tilde{g}_k}^2(y_k, \partial B_k) = d_{g_k}^2(y_k, \partial B_k) \mu_k \longrightarrow \infty.$$

Furthermore, for any finite $\rho > 0$ and $z \in B_k$, if $d_{\tilde{g}_k}(z, y_k) = \text{dist}_{\tilde{g}_k}(z, y_k) < \rho$,

$$(2.5) \quad r_h(\tilde{g}_k)(z) \geq \frac{d_{\tilde{g}_k}(z, \partial B_k)}{d_{\tilde{g}_k}(y_k, \partial B_k)} \geq \frac{d_{\tilde{g}_k}(z, \partial B_k)}{d_{\tilde{g}_k}(z, \partial B_k) + d_{\tilde{g}_k}(z, y_k)} \geq \frac{1}{2},$$

$k \gg 1$. Note that, by (2.2), $\text{Vol}_{\tilde{g}_k}(B_{\tilde{g}_k}(z, \rho)) \geq \kappa \rho^{2n}$ where $z \in B_k$, and $\rho < \mu_k^{\frac{1}{2}} \min\{1, C^{-\frac{1}{2}}\}$. Thus, by Lemma 2.1 and Remark 2.4 in [2], a subsequence of (B_k, \tilde{g}_k, y_k) converges to a complete $C^{1,\alpha}$ -Riemannian manifold (N, g_∞, y_∞) in the $C^{1,\alpha'}$ -sense, $\alpha' < \alpha$, i.e. for any $\bar{r} > 0$ and $k \gg 1$, there is a smooth embedding $F_{\bar{r},k}^* : B_{g_\infty}(y_\infty, \bar{r} + 1) \longrightarrow B_k$ such that $F_{\bar{r},k}^* \tilde{g}_k$ converges to g_∞ in the $C^{1,\alpha'}$ sense. Furthermore,

$$(2.6) \quad r_h(g_\infty)(y_\infty) \leq \liminf_{k \rightarrow \infty} r_h(\tilde{g}_k)(y_k) = 1$$

(c.f. [2]), and

$$(2.7) \quad \text{Vol}_{g_\infty}(B_{g_\infty}(y_\infty, \rho)) \geq \kappa \rho^{2n}$$

for any $\rho > 0$ by (2.2).

Claim 2.2. g_∞ is a $L^{2,n}$ -metric on N , and a subsequence of $F_{\bar{r},k}^* \tilde{g}_k$ converges to g_∞ in the weak $L^{2,n}$ -topology. Thus,

$$(2.8) \quad \int_N |Rm(g_\infty)|^n dv_\infty \leq \sup_k \int_{B_{g_k}(x_k, 2r)} |Rm(g_k)|^n dv_k \leq \varepsilon.$$

Proof. Recall that, for a Riemannian metric g , the Ricci curvature in harmonic coordinates is given

$$g^{ij} \frac{\partial^2 g_{hl}}{\partial x_i \partial x_j} + \mathcal{Q}\left(\frac{\partial g_{rs}}{\partial x_m}\right) = (Ric(g))_{hl},$$

(c.f. [2]) where \mathcal{Q} is a quadratic term. Note that this equation is a uniformly elliptic system of P.D.E. with a uniform $C^{1,\alpha'}$ -bound on the coefficients g^{ij} , and a $C^{0,\alpha'}$ -bound on the term \mathcal{Q} . If there is a L^n -bound on the right side, then the elliptic regularity theory gives a uniform bound on $\|g\|_{L^{2,n}}$ (c.f. [10]). Thus the conclusion follows. \square

Note that, for any $\bar{r} > 0$, there is an $r_0 > 0$ such that, for $z \in B_{g_\infty}(y_\infty, \bar{r} + 1)$ and any domain $\Omega \subset B_{g_\infty}(z, r_0)$, $\text{Vol}_{g_\infty}(\partial\Omega)^{2n} \geq (1 - \delta)c_{2n}\text{Vol}_{g_\infty}(\Omega)^{2n-1}$, where δ is the constant in Theorem 10.1 in [13] or Theorem 29.1 of [12], and c_{2n} is the Euclidean isoperimetric constant. Thus, from the convergence, for any $z' \in B_{\tilde{g}_k}(y_k, \bar{r} + 1)$ and any domain $\Omega \subset B_{\tilde{g}_k}(z', r_0)$, $\text{Vol}_{\tilde{g}_k}(\partial\Omega)^{2n} \geq (1 - \delta)c_{2n}\text{Vol}_{\tilde{g}_k}(\Omega)^{2n-1}$, when $k \gg 1$.

Lemma 2.3. g_∞ is a complete Ricci flat metric on N .

Proof. Note that $\tilde{g}_k(t) = \mu_k g(t_k + \mu_k^{-1}t)$, $t \in [0, \infty)$, are solutions of the normalized Kähler-Ricci flow

$$\partial_t \tilde{g}_k(t) = -\text{Ric}(\tilde{g}_k(t)) + \frac{1}{\mu_k} \tilde{g}_k(t)$$

with initial metrics \tilde{g}_k , which satisfy

$$(2.9) \quad |R(\tilde{g}_k(t))| = \mu_k^{-1} |R(g(t_k + \mu_k^{-1}t))| \leq \frac{C}{\mu_k} \longrightarrow 0, \quad \text{by (2.1),}$$

when $k \longrightarrow \infty$. If $\bar{t} = \mu_k(1 - e^{-\frac{t}{\mu_k}})$ and $\bar{g}_k(\bar{t}) = e^{-\frac{t}{\mu_k}} \tilde{g}_k(t)$, then $\bar{g}_k(\bar{t})$, $\bar{t} \in [0, \mu_k)$ is a solution of the Kähler-Ricci flow

$$\partial_{\bar{t}} \bar{g}_k(\bar{t}) = -\text{Ric}(\bar{g}_k(\bar{t}))$$

on M with initial metric \tilde{g}_k . By Theorem 10.1 in [13] or Theorem 29.1 in [12], there is an $\epsilon > 0$ such that

$$(2.10) \quad |\text{Rm}(\bar{g}_k(\bar{t}))|(z) \leq \frac{1}{\bar{t}} + (\epsilon r_0)^{-2},$$

$0 < \bar{t} < (\epsilon r_0)^2 < 1$, for all $z \in B_{\tilde{g}_k}(y_k, \bar{r} + 1)$. By (2.9),

$$(2.11) \quad |R(\bar{g}_k(\bar{t}))| = \frac{1}{1 - \frac{\bar{t}}{\mu_k}} |R(\tilde{g}_k(t))| \leq \frac{2C}{\mu_k} \longrightarrow 0,$$

$\bar{t} \in [0, (\epsilon r_0)^2]$, when $k \longrightarrow \infty$.

By (2.2), it is easy to see that $\bar{g}_k(\bar{t})$ is κ -noncollapsed, and, for any $z \in B_{\tilde{g}_k}(y_k, \bar{r} + 1)$, the injectivity radius $\text{inj}_{\bar{g}_k((\epsilon r_0)^2)}(z) \geq \iota$ for a positive constant ι independent of k by (2.10). From the compactness theorem for Ricci flow (c.f. Appendix E in [12] or [11]), by passing to a subsequence, $(B_{\tilde{g}_k}(y_k, \bar{r} - 1), \bar{g}_k(\bar{t}), y_k)$, $\bar{t} \in (0, T]$, C^∞ -converges to $(B_\infty, g_\infty(\bar{t}), y_\infty)$, $\bar{t} \in (0, T]$, where $g_\infty(\bar{t})$ is a solution of Ricci flow on B_∞ , and $T < (\epsilon r_0)^2$. By (2.11), $|R(g_\infty(\bar{t}))| \equiv 0$. This implies that $|\text{Ric}(g_\infty(\bar{t}))| \equiv 0$, and $g_\infty(\bar{t}) \equiv g_\infty(T)$ is a Ricci-flat metric on B_∞ . By Theorem 36.2 in [12],

$$d_{GH}((B_\infty, g_\infty(T)), (B_{g_\infty}(y_\infty, \bar{r}-1), g_\infty)) = \lim_{\bar{t} \rightarrow 0} d_{GH}((B_\infty, g_\infty(\bar{t})), (B_{g_\infty}(y_\infty, \bar{r}-1), g_\infty)) = 0,$$

where d_{GH} denotes the Gromov-Hausdorff distance. By letting $\bar{r} \rightarrow \infty$ and taking a diagonalized sequence, we obtain that g_∞ is a Ricci flat metric on N . \square

Lemma 2.3 and (2.7) imply that there is a global bound for the Sobolev constant on (N, g_∞) (c.f. [8] or [1]). Let ε be the corresponding constant in Lemma 2.1 of [1], which depends on the Sobolev constant on (N, g_∞) . By Claim 2.2, Lemma 2.3 and Lemma 2.1 in [1],

$$\sup_{B_{g_\infty}(y_\infty, \frac{s}{2})} |Rm(g_\infty)| \leq \frac{C'}{s^2} \longrightarrow 0, \quad \text{as } s \rightarrow \infty,$$

and, thus, g_∞ is a flat metric. By (2.7), (N, g_∞) is the standard Euclidean space \mathbb{R}^{2n} (c.f. [1]). It contradicts to (2.6), since the harmonic radius of \mathbb{R}^{2n} is infinite. We obtain the conclusion. \square

3. PROOF OF THEOREM 1.1

Let $g(t)$, $t \in [0, +\infty)$, be a solution of the normalized Kähler-Ricci flow (1.1) on a compact Kähler n -manifold M with $c_1(M) > 0$ such that $g(0) \in 2\pi c_1(M)$, and

$$(3.1) \quad \int_M |Rm(g(t))|^n dv_t \leq C,$$

where C is a constant independent of t . Assume that $t_k \rightarrow \infty$ is a sequence of times.

Lemma 3.1. *If there is a constant $\mathcal{V} > 0$ independent of t such that*

$$(3.2) \quad Vol_{g(t)}(B_{g(t)}(x, r)) \leq \mathcal{V}r^{2n},$$

for any $r \leq 1$ and $x \in M$, then, by passing to a subsequence, $(M, g(t_k))$ converges to an orbifold (X, h) with only finitely many singular points $\{q_j\}$ in the Gromov-Hausdorff topology, where h is a C^0 -orbifold metric, and is $C^{1,\alpha}$ off the singular points. Furthermore, for any compact subset $K \subset X \setminus \{q_j\}$, there are smooth embeddings $F_{K,k} : K \rightarrow M$ such that $F_{K,k}^ g(t_k)$ converges to h in the $C^{1,\alpha'}$ (resp. weak $L^{2,n}$) topology, $\alpha' < \alpha$.*

Proof. By (2.2) and (3.1), all hypothesis of Theorem 1.1 in [3] are satisfied except (1.5) in [3], i.e. small curvature estimate, which is used in the arguments in Section 2.1 of [3] for obtaining the local $C^{1,\alpha}$ -convergence, and weak $L^{2,p}$ -convergence. However, we have Proposition 2.1, which gives a lower bound for harmonic radius, when the n -norm of curvature is small enough. This is enough to obtain the local $C^{1,\alpha'}$ -convergence (c.f. [2]), and weak $L^{2,n}$ -convergence by the proof of Claim 2.2. Thus, by (3.2), the arguments in Section 2.1 of [3] and the proof of Theorem 2.6 in [2], we obtain that, by passing to a subsequence, $(M, g(t_k))$ converges to a metric space (X, h) in the Gromov-Hausdorff topology. Here X is a multi-fold with a finite set $S = \{q_j\}$ of singular points in the sense of [21], i.e. X is a smooth manifold off a finite set of singular points S , and a neighborhood of each singular point q_j is a finite union of cones on spherical space forms $C(S^{2n-1}/\Gamma)$, $\Gamma \subset SO(2n)$, where the vertex of each cone is identified with the point q_j . The metric h is a $C^{1,\alpha} \cap L^{2,n}$ -Riemannian metric on $X \setminus S$, and, in a local uniformization $B^{2n} \setminus \{0\}$, h extends to a C^0 -metric on the ball B^{2n} . Furthermore, for any compact subset $K \subset X \setminus S$, there are smooth embeddings $F_{K,k} : K \rightarrow M$ such that $F_{K,k}^* g_k$ converges to

h in the $C^{1,\alpha'}$ and weak $L^{2,n}$ topologies, $\alpha' < \alpha$, where $g_k = g(t_k)$. Thus the only thing we are supposed to prove is that there is exactly one cone at each singular point.

From the arguments in Section 2.1 of [3], for any singular point $x \in S$, there is a sequence of points $x_k \in (M, g_k)$ such that $x_k \rightarrow x$, when $k \rightarrow \infty$, and, for any $r > 0$,

$$(3.3) \quad \liminf_{k \rightarrow \infty} \int_{B_{g_k}(x_k, r)} |Rm(g_k)|^n dv_k \geq \varepsilon,$$

where ε is the constant in Proposition 2.1. If there is a $x \in S$ with more than one cones attaching it, we imitate the proof in [21] to obtain a contradiction. Choose a radius $\bar{r} > 0$ small enough, and a sequence of points $x_k \in (M, g_k)$ such that the harmonic radius satisfies

$$(3.4) \quad \inf_{B_{g_k}(x_k, \bar{r})} i_h(g_k) = i_h(g_k)(x_k) \rightarrow 0, \quad \text{as } k \rightarrow \infty,$$

and

$$(3.5) \quad \int_{B_h(x, \bar{r})} |Rm(h)|^n dv_h \leq \frac{\varepsilon}{2},$$

and $B_h(x, \bar{r}) \setminus \{x\}$ has several components. We choose $r_k < \bar{r}$ such that

$$(3.6) \quad \int_{D_{g_k}(r_k, \bar{r})} |Rm(g_k)|^n dv_k = \varepsilon,$$

where $D_{g_k}(r_k, \bar{r}) = B_{g_k}(x_k, \bar{r}) \setminus B_{g_k}(x_k, r_k)$. Note that the annulus $D_{g_k}(r_k, \bar{r})$ has several components.

By (3.4) and (3.5), it is easy to see that $x_k \rightarrow x$ and $r_k \rightarrow 0$ when $k \rightarrow \infty$. From the same arguments as above, a subsequence of $(M, r_k^{-2}g_k, x_k)$ converges to a complete multi-fold $(N_1, g_{1,\infty}, x_{1,\infty})$ with a finite set S_1 of singular points in the pointed Gromov-Hausdorff sense, where $g_{1,\infty}$ is a $C^{1,\alpha} \cap L^{2,n}$ -Riemannian metric on $N_1 \setminus S_1$. And, for any compact subset $K \subset N_1 \setminus S_1$, there are smooth embeddings $F_{1,K,k} : K \rightarrow M$ such that $F_{1,K,k}^* r_k^{-2} g_k$ converges to $g_{1,\infty}$ in the $C^{1,\alpha'}$ and weak $L^{2,n}$ topologies, $\alpha' < \alpha$. By (3.6), there is no singular points in $N_1 \setminus B_{g_{1,\infty}}(x_{1,\infty}, 1)$. By considering the rescaled Ricci flow $r_k^{-2} g(t_k + r_k^2 t)$, $t \in [0, \infty)$, the arguments in the proof of Lemma 2.3 imply that $g_{1,\infty}$ is a Ricci-flat metric on $N_1 \setminus S_1$.

If, for any $x_1 \in S_1$, there is only one cone at x_1 , i.e. N_1 is an orbifold, then $g_{1,\infty}$ is a Ricci flat orbifold metric on N_1 by (3.1) and [1], i.e. in a local uniformization $B^{2n} \setminus \{x_1\}$, $g_{1,\infty}$ extends to a Ricci flat metric on the ball B^{2n} . From the construction above, N_1 has several ends, i.e. $N_1 \setminus B_{g_{1,\infty}}(x_{1,\infty}, 1)$ is not connected. By the convergence and the Perelman's estimate (2.2), there is a $\kappa > 0$ such that, for any $y_1 \in N_1$ and $\rho > 0$,

$$\text{Vol}_{g_{1,\infty}}(B_{g_{1,\infty}}(y_1, \rho)) \geq \kappa \rho^{2n}.$$

This implies that there is a global bound for the Sobolev constant on $N_1 \setminus B_{g_{1,\infty}}(x_{1,\infty}, 1)$ (c.f. [8] or [1]). Thus, by Lemma 2.1 of [1], the small curvature estimate (1.5) in [3] is satisfied. Theorem 1.2 and Remark 2.8 of [3] show that $(N_1, g_{1,\infty}, x_{1,\infty})$ is an Asymptotically Locally Euclidean multi-fold with several ends. Each end E_j of N_1 is diffeomorphic to $(\mathbb{R}^{2n} \setminus B^{2n})/\Gamma_j$, where Γ_j is a finite subgroup of $SO(2n)$ acting on S^{2n-1} .

freely. However, since the number of ends of N_1 is larger than one, the splitting theorem for orbifolds (c.f. [5]) implies that $(N_1, g_{1,\infty})$ is isometric to $Y \times \mathbb{R}^1$ where Y is a complete Ricci-flat orbifold. It is a contradiction. Thus there is a $x_1 \in S_1$ such that there are more than one cones at x_1 .

Now we do the same process as above for N_1 and x_1 , and obtain a multi-fold N_2 with finite singular set S_2 . If there is only one cone at each point of S_2 , then we stop, and obtain a contradiction as above. Otherwise we repeat the procedure. This process must terminate in finite steps, since each singularity takes at least ε of curvature by the construction (c.f. [21]). Finally, we shall obtain a contradiction. Thus X is an orbifold. \square

Lemma 3.2. *There is a constant $\mathcal{V} > 0$ independent of t such that*

$$(3.7) \quad \text{Vol}_{g(t)}(B_{g(t)}(x, r)) \leq \mathcal{V} r^{2n},$$

for any $r \leq 1$ and $x \in M$.

Proof. If it is not true, there exists a sequence of times t_i , and a sequence of balls $B_{g_i}(x_i, r_i)$ such that

$$\frac{\text{Vol}_{g_i}(B_{g_i}(x_i, r_i))}{r_i^{2n}} \geq \mathcal{V}_i \longrightarrow \infty,$$

as $i \rightarrow \infty$, where $g_i = g(t_i)$. Let s_i be the smallest radius such that, for some $y_i \in (M, g_i)$,

$$\text{Vol}_{g_i}(B_{g_i}(y_i, s_i)) \geq 2\omega_{2n}s_i^{2n},$$

where ω_{2n} is the volume of 1-ball in the Euclidean space \mathbb{R}^{2n} . (Note that, for any $x \in M$, $\text{Vol}_{g_i}(B_{g_i}(x, r)) \sim \omega_{2n}r^{2n}$ when $r \rightarrow 0$.) From the arguments in the proof of Theorem 1.1 in [3], we can assume that $s_i \rightarrow 0$ as $i \rightarrow \infty$.

If $\tilde{g}_i = s_i^{-2}g_i$, then, for any $r < 1$ and $x \in M$,

$$(3.8) \quad \text{Vol}_{\tilde{g}_i}(B_{\tilde{g}_i}(x, r)) \leq 2\omega_{2n}r^{2n}, \quad \text{and} \quad \text{Vol}_{\tilde{g}_i}(B_{\tilde{g}_i}(y_i, 1)) = 2\omega_{2n}$$

by the choice of s_i . By the arguments in Section 2.1 in [3] and the proof of Lemma 3.1, a subsequence of (M, \tilde{g}_i, y_i) converges to a complete orbifold $(N, \tilde{g}_\infty, y_\infty)$ with only finite many singular points $\{q_j\}$, where \tilde{g}_∞ is a C^0 -orbifold metric, and is $C^{1,\alpha}$ off the singular points. Furthermore, for any compact subset $K \subset N \setminus \cup_j \{q_j\}$, there are smooth embeddings $F_{K,i} : K \rightarrow M$ such that $F_{K,i}^*\tilde{g}_i$ converges to \tilde{g}_∞ in the $C^{1,\alpha'}$ and weak $L^{2,n}$ topologies, $\alpha' < \alpha$. By considering the re-scaled Kähler-Ricci flow $\tilde{g}_k(t) = s_i^{-2}g(t_i + s_i^2t)$, $t \in [0, \infty)$, and the proof of Lemma 2.3, we obtain that \tilde{g}_∞ is a Ricci flat metric on $N \setminus \cup_j \{q_j\}$. From the weak $L^{2,n}$ -convergence,

$$\int_{N \setminus \cup_j \{q_j\}} |Rm(\tilde{g}_\infty)|^n dv_\infty \leq C < \infty.$$

Thus \tilde{g}_∞ can be extended to a Ricci flat orbifold metric on N (c.f. [1]). Note that volume comparison theorem holds also for orbifolds by [4]. Thus, for any $r > 0$, we obtain that

$$\text{Vol}_{\tilde{g}_\infty}(B_{\tilde{g}_\infty}(x, r)) \leq \omega_{2n}r^{2n}.$$

By (3.8) and the $C^{1,\alpha'}$ -convergence, we obtain $Vol_{\tilde{g}_\infty}(B_{\tilde{g}_\infty}(y_\infty, 1)) = 2\omega_{2n}$, which is a contradiction. \square

Proof of Theorem 1.1. By Lemmas 3.1 and 3.2, a subsequence of $(M, g(t_k))$ converges to an orbifold (X, h) with only finite many singular points $\{q_j\}$ in the Gromov-Hausdorff topology, where h is a C^0 -orbifold metric, and is $C^{1,\alpha}$ off the singular points. Furthermore, for any $r > 0$, there are smooth embeddings $F_{r,k} : X \setminus \cup_j B_h(q_j, r) \rightarrow M$ such that $F_{r,k}^* g(t_k)$ converges to h in the $C^{1,\alpha'}$ and weak $L^{2,n}$ topologies, $\alpha' < \alpha$. Actually, h is a Kähler metric (c.f. [15]). By the same arguments as in the proof of Proposition 2.1, for $k \gg 1$, there is a r_0 independent of k such that, for any domain $\Omega \subset B_{g_k}(z, r_0)$, $z \in X \setminus \cup_j B_h(q_j, 2r)$, $Vol_{g_k}(\partial\Omega)^{2n} \geq (1 - \delta)c_{2n}Vol_{g_k}(\Omega)^{2n-1}$, where $g_k = g(t_k)$, δ is the constant in Theorem 29.1 of [12], and c_{2n} is the Euclidean isoperimetric constant.

If $\tilde{t} = 1 - e^{-t}$ and $\tilde{g}_k(\tilde{t}) = e^{-t}g_k(t)$, where $g_k(t) = g(t_k + t)$, then $\tilde{g}_k(\tilde{t})$, $\tilde{t} \in [0, 1)$ is a solution of the Kähler-Ricci flow

$$\partial_{\tilde{t}}\tilde{g}_k(\tilde{t}) = -Ric(\tilde{g}_k(\tilde{t}))$$

on M with initial metric g_k . By Theorem 10.1 in [13] or Theorem 29.1 in [12], there is a $\epsilon > 0$ such that

$$|Rm(\tilde{g}_k(\tilde{t}))|(z) \leq \frac{1}{\tilde{t}} + (\epsilon r_0)^{-2},$$

for $0 < \tilde{t} < (\epsilon r_0)^2 < 1$, and all $z \in X \setminus \cup_j B_h(q_j, 2r)$. Since $\tilde{g}_k(\tilde{t})$ differs to $g_k(t)$ only by rescalings in space and time, i.e. $g_k(t) = \frac{1}{1-\tilde{t}}\tilde{g}_k(\tilde{t})$, where $t = \log(\frac{1}{1-\tilde{t}})$, we obtain

$$(3.9) \quad |Rm(g_k(t))|(z) \leq \frac{1}{e^{-t}(1 - e^{-t})} + e^t(\epsilon r_0)^{-2},$$

for all $z \in X \setminus \cup_j B_h(q_j, 2r)$ and $0 < t < -\log(1 - (\epsilon r_0)^2)$. By (2.2) and (3.9), $g_k(t)$ is κ -noncollapsed, and, for any $z \in X \setminus \cup_j B_h(q_j, 2r)$, the injectivity radius $\text{inj}_{\tilde{g}_k(t_0)}(z) \geq \iota$, where $t_0 = -\frac{1}{2}\log(1 - (\epsilon r_0)^2)$, for a constant ι independent of k . By the compactness theorem for Ricci flow (c.f. Appendix E in [12] or [11]), by passing to a subsequence, $(X \setminus \cup_j B_h(q_j, 4r), g_k(t))$, $t \in (0, T]$, converges to $(X \setminus \cup_j B_h(q_j, 4r), h(t))$, $t \in (0, T]$, where $h(t)$ is a solution of Ricci flow on $X \setminus \cup_j B_h(q_j, 4r)$, and $T < t_0$. By Theorem 36.2 in [12],

$$(3.10) \quad \lim_{t \rightarrow 0} d_{GH}((X \setminus \cup_j B_h(q_j, 4r), h), (X \setminus \cup_j B_h(q_j, 4r), h(t))) = 0.$$

From the arguments in the proof of Theorem 12 in [19], $h(t)$, $t \in (0, T]$, satisfies the Kähler-Ricci soliton equation, i.e. there are smooth functions $f(t)$, $t \in (0, T]$, such that

$$Ric(h(t)) - h(t) = \nabla \bar{\nabla} f(t),$$

and

$$\nabla \nabla f(t) = \bar{\nabla} \bar{\nabla} f(t) = 0.$$

Note that there is a fixed vector field V on $X \setminus \cup_j B_h(q_j, 4r)$ such that

$$h(t) = \phi^{-1}(t)^* h(t_1), \quad \text{and} \quad f(t) = \phi^{-1}(t)^* f(t_1),$$

where $\{\phi(t)\}$ is the 1-parameter group of diffeomorphisms generated by $-V$, and $\phi(t_1) = id$ (c.f. Appendix C of [12]). By letting $0 < t_1 \ll 1$ such that $\phi^{-1}(0)(X \setminus \cup_j B_h(q_j, 8r)) \subset$

$X \setminus \cup_j B_h(q_j, 4r)$, we obtain that $\phi^{-1}(0)^*h(t_1)$ is a metric satisfying the Kähler-Ricci soliton equation on $X \setminus \cup_j B_h(q_j, 8r)$, and

$$\lim_{t \rightarrow 0} d_{GH}((X \setminus \cup_j B_h(q_j, 8r), \phi^{-1}(0)^*h(t_1)), (X \setminus \cup_j B_h(q_j, 8r), h(t))) = 0.$$

Thus, by (3.10), $(X \setminus \cup_j B_h(q_j, 8r), h)$ is isometric to $(X \setminus \cup_j B_h(q_j, 8r), \phi^{-1}(0)^*h(t_1))$. By letting $r \rightarrow 0$ and taking a diagonalized sequence, we obtain that h satisfies the Kähler-Ricci soliton equation on $X \setminus \cup_j \{q_j\}$. □

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